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NON-HOLONOMIC SYSTEMS IN VIEW OF HAMILTONIAN PRINCIPLE

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PRESENTATION OUTLINE

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1. INTRODUCTORY REMARKS

1.1. Physical and engineering motivation

Dynamic systems modeling:

- classical mechanics, applications in engineering,
- control of mobile robots, aero, naval and space vehicles,
- modern approach of electromagnetism,
- fluid mechanics, aeroelasticity, fluid-structure interaction,
- dynamics of multi-phase mixtures (suspension, aerosol, magnetic fluid, etc),
- climatology and meteorology,
- description of relativistic particles with spin,
- formulation of string theory,
- chemical processes and reactors,
- neuro-physiology and other medical disciplines,
- biomechanics and life sciences,
- social processes: economy, sociology, psychology, management, etc.
- transportation, logistics.

1.2. View of classical dynamics and related disciplines

Basic principles:

- Hamiltonian principle and Lagrangian formalism,
- Appell - Gibbs function,
- d'Alembert, Jourdain, Gauss, Appell-Chetayev, etc.

Higher Dynamic Systems:

- systems with non-holonomic constraints of higher order (linear or nonlinear),
- one-sided and other non-continuous links,
- indefinite description of the system,
- higher order time derivatives in energy functional,
- requests on the system movement and trajectory character,
- trajectory planing and on-line control,
- stability testing, post-critical response nearby bifurcation points and homo-/hetero-clinic orbits.

2. BASIC CONSIDERATIONS

2.1. Holonomic and non-holonomic constraints

Holonomic constraints:

linear: $\boldsymbol{\Psi}(t) \cdot \mathbf{u}(t) = 0, \quad \boldsymbol{\Psi}(t) \in \mathbb{R}^{l,n}, \quad \mathbf{u}(t) \in \mathbb{R}^n$

nonlinear: $\boldsymbol{\psi}(\mathbf{u}, t) = 0, \quad \boldsymbol{\psi}(\mathbf{u}, t) \in \mathbb{R}^l, \quad 0 \le l \le n,$

Non-holonomic constraints of the first order:

linear: $\boldsymbol{\Phi}(\mathbf{u}, t) \cdot \dot{\mathbf{u}}(t) + \boldsymbol{\chi}(\mathbf{u}, t) = 0, \quad \boldsymbol{\Phi}(\mathbf{u}, t) \in \mathbb{R}^{l,n}, \quad \boldsymbol{\chi}(\mathbf{u}, t) \in \mathbb{R}^l.$

nonlinear: $\boldsymbol{\varphi}(\mathbf{u}, \dot{\mathbf{u}}, t) = 0, \quad \boldsymbol{\varphi}(\mathbf{u}, t) \in \mathbb{R}^l, \quad 0 \le l \le n.$

n - number of degrees of freedom,
l - number of constraints,
$k = n - l$ - effective number of degrees of freedom.

Constraints differentiation → linear constraints of the 2nd order:

$$\boldsymbol{\varphi}^*(\mathbf{u}, \dot{\mathbf{u}}, t) \cdot \ddot{\mathbf{u}} + \dot{\boldsymbol{\varphi}}(\mathbf{u}, \dot{\mathbf{u}}, t) = 0 \qquad \boldsymbol{\varphi}^*(\mathbf{u}, \dot{\mathbf{u}}, t) \in \mathbb{R}^{l,n} \quad 0 \le l \le n,$$

$\boldsymbol{\varphi}^*(\mathbf{u}, \dot{\mathbf{u}}, t)$ - matrix of total derivatives of the vector $\boldsymbol{\varphi}(\mathbf{u}, \dot{\mathbf{u}}, t)$
$\dot{\boldsymbol{\varphi}}(\mathbf{u}, \dot{\mathbf{u}}, t)$ - scalar time derivatives of the vector $\boldsymbol{\varphi}(\mathbf{u}, \dot{\mathbf{u}}, t)$

Non-holonomic constraints of the $q-th$ order:

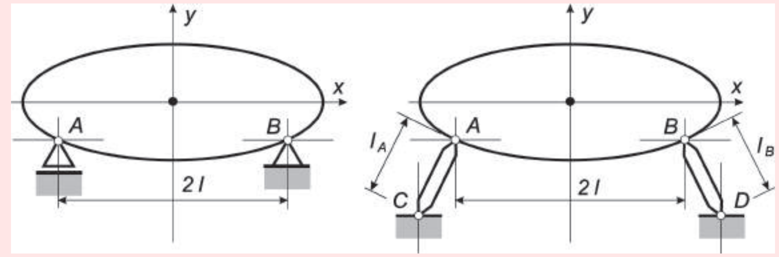
linear: $\boldsymbol{\Phi}(\mathbf{u}, .., \mathbf{u}^{(q-1)}, t) \cdot \mathbf{u}^{(q)} + \boldsymbol{\chi}(\mathbf{u}, .., \mathbf{u}^{(q-1)}, t) = 0, \quad (\mathbf{u}^{(1)} = \dot{\mathbf{u}})$

nonlinear: $\boldsymbol{\varphi}(\mathbf{u}, \mathbf{u}^{(1)}, .., \mathbf{u}^{(q)}, t) = 0,$

q - highest derivative in constraints; q can be higher than the order of movement equations
$\mathbf{u}^{(1)}, .., \mathbf{u}^{(q)})$ - 1st,.., q-th derivative with respect to time; defined on $2n+1$ manifold ,

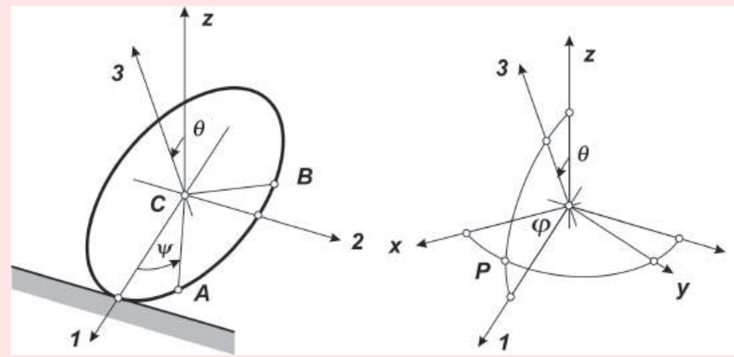
Constraints differentiation → linear constraints of the $q+1$st order:

$$\boldsymbol{\varphi}^{*T}(\mathbf{u}, \mathbf{u}^{(1)}, .., \mathbf{u}^{(q)}, t) \cdot \mathbf{u}^{(q+1)} + \dot{\boldsymbol{\varphi}}(\mathbf{u}, \mathbf{u}^{(1)}, .., \mathbf{u}^{(q)}, t) = 0, \ \boldsymbol{\varphi}^*(\mathbf{u}, \dot{\mathbf{u}}, t) \in \mathbb{R}^{l,n} \quad 0 \le l \le n$$



$u_y(A) = 0, \qquad u_x^2(A) + u_y^2(A) = l_A^2,$

$u_x(B) = 0, u_y(B) = 0. \qquad u_x^2(B) + u_y^2(B) = l_B^2,$



Some domains of occurrence:
- special problems of classical mechanics,
- robotics (mobile robots),
- control of trajectories (space and naval engineering),
- neuro-physiology, life sciences
- particle physics,
- non-smooth and discontinuous constraints,
- one-sided variations,
- iso-perimetric problems, reflective extremals.
- and many others.

2.2. Definition of a holonomic and non-holonomic system

Holonomic, 1st order non-holonomic constraints - Appell & Chetayev definition:

$$\boldsymbol{\Phi}(\mathbf{u},t)\delta\mathbf{u} \;=\; 0, \quad \text{resp.} \qquad \nabla_{\mathbf{u}^{(1)}}\boldsymbol{\varphi}^T(\mathbf{u},\mathbf{u}^{(1)},t)\delta\mathbf{u} \;=\; 0,$$

$\delta\mathbf{u} = [\delta u_1, .., \delta u_n]^T$ – **vector of virtual displacements**

$\nabla_{\mathbf{u}^{(1)}} = [\partial/\partial u_1^{(1)}, .., \partial/\partial u_n^{(1)}]^T$ – **1st derivatives operator with respect to velocities** $u_1^{(1)}, .., u_n^{(1)}$

$q-th$ order nonholonomic constraints - Appell & Chetayev definition:

$$\boldsymbol{\Phi}(\mathbf{u},..,\mathbf{u}^{(q-1)},t)\delta\mathbf{u} \;=\; 0, \quad \text{resp.} \quad \nabla_{\mathbf{u}^{(q)}}\boldsymbol{\varphi}^T(\mathbf{u},..,\mathbf{u}^{(q)},t)\delta\mathbf{u} \;=\; 0.$$

Appell-Chetayev system; guarancy of resluts equivalence (d'Alembert, Jourdain, Gauss, etc.)

Hamiltonian functional:

$$\delta\mathcal{S}\{\mathbf{u}\} \;=\; \int_0^{t_1} \delta\,(\mathcal{T}(t) - \mathcal{V}(t))\mathrm{d}t \;=\; 0$$

$\mathcal{T}(t) \;=\; \mathcal{T}(\mathbf{u},\dot{\mathbf{u}},..,\mathbf{u}^{(p)},t)$ – **kinetic energy**, $\mathcal{V}(t) \;=\; \mathcal{V}(\mathbf{u},t)$, $\dot{\mathbf{u}} \;=\; \mathbf{u}^{(1)}, ...$ – **potential energy**.

Assumption:

$$\mathbf{u}^*(t,\kappa) \;=\; \mathbf{u}(t) + \kappa\cdot(\tilde{\mathbf{u}}(t) - \mathbf{u}(t)) \;=\; \mathbf{u}(t) + \kappa\cdot\delta\mathbf{u}(t),$$

Approximation:

$$\delta\mathcal{S}\{\mathbf{u}\} = \lim_{\kappa=0}\left[\frac{\mathrm{d}}{\mathrm{d}\kappa}\int_0^{t_1}\Big(\mathcal{T}\left(\mathbf{u}(t,\kappa),\mathbf{u}^{(1)}(t,\kappa),..,\mathbf{u}^{(p)}(t,\kappa)\right) - \mathcal{V}(\mathbf{u}(t,\kappa))\Big)\mathrm{d}t\right],$$

$$\delta\mathcal{S}\{\mathbf{u}\} = \int_0^{t_1}\left[\nabla^T_{\mathbf{u}^{(p)}}\mathcal{T}(t)\delta\mathbf{u}^{(p)} +, .., + \nabla^T_{\mathbf{u}^{(1)}}\mathcal{T}(t)\delta\mathbf{u}^{(1)} + \nabla^T_{\mathbf{u}}\left(\mathcal{T}(t)\delta\mathbf{u} - \mathcal{V}(t)\right)\delta\mathbf{u}\right]\mathrm{d}t = 0,$$

$$\delta\mathcal{S}\{\mathbf{u}\} = \int_0^{t_1}\left[(-1)^p\frac{\mathrm{d}^p}{\mathrm{d}t^p}(\nabla^T_{\mathbf{u}^{(p)}}\mathcal{T}(t)), .., -\frac{\mathrm{d}}{\mathrm{d}t}(\nabla^T_{\mathbf{u}^{(1)}}\mathcal{T}(t)) + \nabla^T_{\mathbf{u}}\left(\mathcal{T}(t) - \mathcal{V}(t)\right)\right]\delta\mathbf{u}\mathrm{d}t = 0,$$

$$(-1)^p\frac{\mathrm{d}^p}{\mathrm{d}t^p}(\nabla^T_{\mathbf{u}^{(p)}}\mathcal{T}(t)), .., -\frac{\mathrm{d}}{\mathrm{d}t}(\nabla^T_{\mathbf{u}^{(1)}}\mathcal{T}(t)) + \nabla^T_{\mathbf{u}}\left(\mathcal{T}(t) - \mathcal{V}(t)\right) \;=\; 0,.$$

n **equations;** l **constraints reduce the effective number of independent components** $k = n - l$.

2.3. Assembling of the governing system in holonomic and non-holonomic versions

Independence of virtual displacements with respect to constraints:

$$\boldsymbol{\nabla}_{\mathbf{u}^{(q)}}\boldsymbol{\varphi}^T(\mathbf{u},..,\mathbf{u}^{(q)},t)\delta\mathbf{u} \;=\; 0 \qquad \Longrightarrow \qquad \mathbf{A}\cdot\delta\mathbf{u} \;=\; 0, \quad \mathbf{A}\in\mathbb{R}^{l,n} \qquad \cdots\cdots\cdots \quad l \;\textbf{equations}$$

Sub-matrix: $\mathbf{A}^l \in \mathbb{R}^{l,l}: \quad \det\mathbf{A}^l \;=\; \begin{vmatrix} a_{11}, & a_{12}, & .., & a_{1l} \\ \vdots & & \ddots & \vdots \\ a_{l1}, & a_{2l}, & .., & a_{ll} \end{vmatrix} \neq 0, \quad \delta u_{l+1},..,\delta u_n$ **arbitrary variations**

Variation of the extended functional:

$$\delta S\{\mathbf{u}\} = \int\limits_0^{t_1} \left[(-1)^p\frac{\mathrm{d}^p}{\mathrm{d}t^p}(\boldsymbol{\nabla}^T_{\mathbf{u}^{(p)}}\mathcal{T}(t)),..,-\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\nabla}^T_{\mathbf{u}^{(1)}}\mathcal{T}(t)) + \boldsymbol{\nabla}^T_{\mathbf{u}}(\mathcal{T}(t)-\mathcal{V}(t)) + \boldsymbol{\nabla}_{\mathbf{u}^{(q)}}\boldsymbol{\varphi}^T\boldsymbol{\lambda}^T\right]\delta\mathbf{u}\mathrm{d}t = 0.$$

Lagrangian equations of motion:

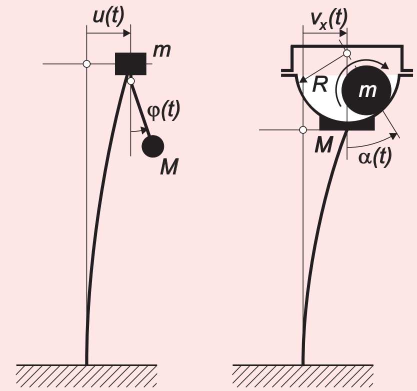
$$(-1)^p\frac{\mathrm{d}^p}{\mathrm{d}t^p}(\boldsymbol{\nabla}^T_{\mathbf{u}^{(p)}}\mathcal{T}(t)),..,-\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\nabla}^T_{\mathbf{u}^{(1)}}\mathcal{T}(t)) + \boldsymbol{\nabla}^T_{\mathbf{u}}\left(\mathcal{T}(t)-\mathbf{I}^T\mathcal{V}(t)\right) \;=\; \mathbf{Q} + \boldsymbol{\nabla}_{\mathbf{u}^{(q)}}\boldsymbol{\varphi}^T\boldsymbol{\lambda}^T \qquad \cdots n \;\textbf{equations}$$

$\mathbf{u} = [u_1,..,u_n] \in \mathbb{R}^n$ - **unknown components of the movement vector,**
$\boldsymbol{\lambda} = \boldsymbol{\lambda}(t) \in \mathbb{R}^l$ - l **unknown Lagrange multipliers,** $\boldsymbol{\lambda}(t)$ **usually forces in reactions,**
$n + l$ **equations together.**

linear constraints - $\boldsymbol{\lambda}$ **can be sometimes eliminated,**
n **equations remain** ($n - l$**: movement,** l**: constraints).**

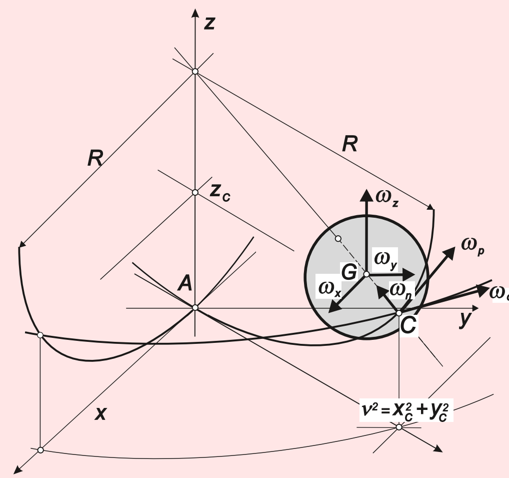
3. DEMONSTRATION OF INDIVIDUAL SYSTEMS

3.1. First order linear constraints



Non-holonomic constraints (Pfaff type):

$$\begin{aligned} \dot{u}_{Cx} &= \omega_y(u_{Cz} - R) - \omega_z u_{Cy}, \\ \dot{u}_{Cy} &= \omega_z u_{Cx} - \omega_x(u_{Cz} - R), \\ \dot{u}_{Cz} &= \omega_x u_{Cy} - \omega_y u_{Cx}, \end{aligned}$$



Equations of movement:

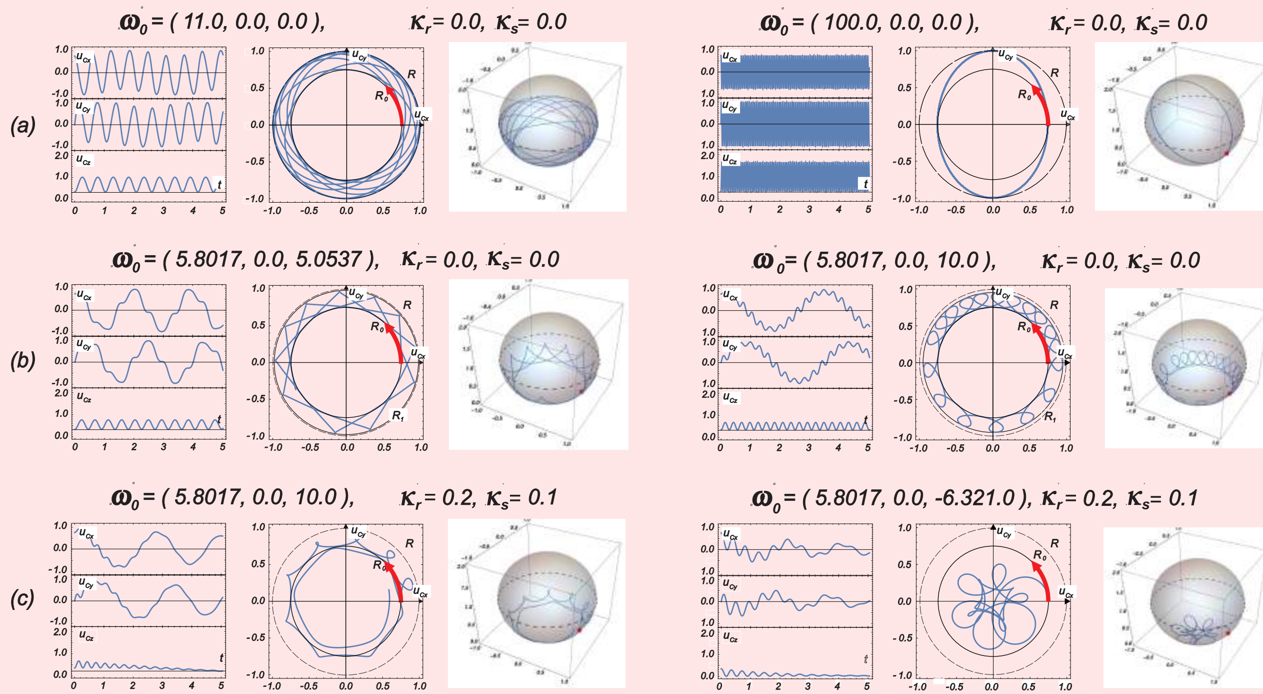
$$\begin{aligned} J_s\dot{\omega}_x &= ((\ddot{u}_{Ay} + \rho(\omega_z\dot{u}_{Cx} - \omega_x\dot{u}_{Cz}))(u_{Cz} - R) - \\ &\qquad -u_{Cy}(g + \rho(\omega_x\dot{u}_{Cy} - \omega_y\dot{u}_{Cx})))/\rho - D_{Gx}/M, \\ J_s\dot{\omega}_y &= (-(\ddot{u}_{Ax} + \rho(\omega_y\dot{u}_{Cz} - \omega_z\dot{u}_{Cy}))(u_{Cz} - R) + \\ &\qquad +u_{Cx}(g + \rho(\omega_x\dot{u}_{Cy} - \omega_y\dot{u}_{Cx})))/\rho - D_{Gy}/M, \\ J_s\dot{\omega}_z &= ((\ddot{u}_{Ax} + \rho(\omega_y\dot{u}_{Cz} - \omega_z\dot{u}_{Cy}))u_{Cy} - \\ &\qquad -(\ddot{u}_{Ay} + \rho(\omega_z\dot{u}_{Cx} - \omega_x\dot{u}_{Cz}))u_{Cx})/\rho - D_{Gz}/M. \end{aligned}$$

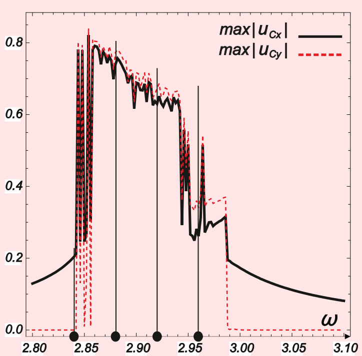
$$J_s = \underbrace{(J + M\rho^2R^2)}_{\text{mass inertia moment of the ball}} \;/\; M\rho^2,$$

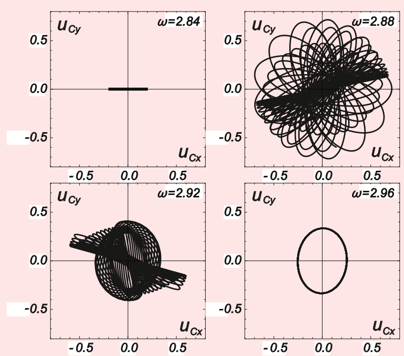
no damping, no spin:

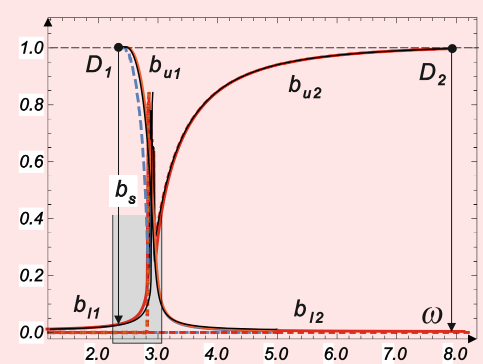
no damping, spin included:

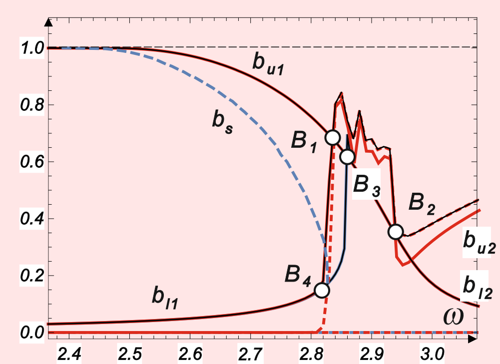
damping and spin included:









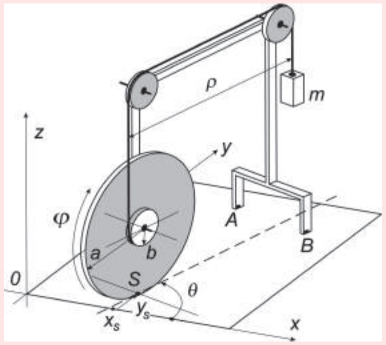


3.2. First order nonlinear constraints

Basic relations:

$$u_x = u_{xs} + \rho\cos\theta, \quad u_y = u_{ys} + \rho\sin\theta, \quad \mathrm{d}u_z = b\mathrm{d}\varphi$$

Appell-Hamel dynamic system.



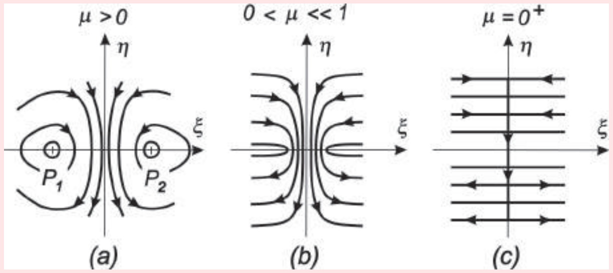
Non-holonomic (Pfaff) constraints:

$$\dot{u}_{xs} = a\dot{\varphi}\cos\theta, \qquad \dot{u}_{ys} = a\dot{\varphi}\sin\theta$$

Kinetic and potential energies:

$$\mathcal{T}(t) = \tfrac{1}{2}m(\dot{u}_x^2 + \dot{u}_y^2 + \dot{u}_z^2) + \tfrac{1}{2}M(\dot{u}_{xS}^2 + \dot{u}_{yS}^2) + \tfrac{1}{2}(J_z\dot{\theta}^2 + J_c\dot{\varphi}^2),$$
$$\mathcal{V}(t) = mg(H_0 - u_z)$$

Trajectories of the contact point - $\mu \geq 0$



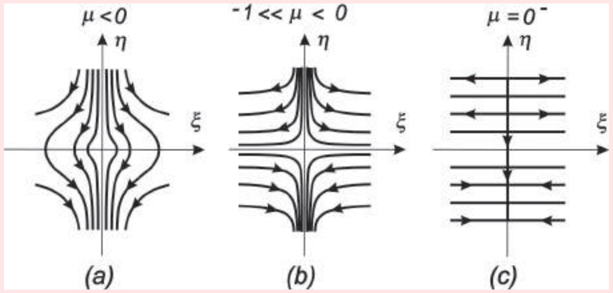
Lagrange equations:

$$(J_z + m\rho^2)\ddot{\theta} + ma\rho\cdot\theta\ddot{\varphi} = 0,$$
$$((m+M)a^2 + mb^2 + J_c)\ddot{\varphi} + ma\rho\cdot\dot{\theta}^2 + mgb = 0.$$

Reduced form $(M = J_z = J_c = 0) \quad \Longrightarrow$**:**

$$\rho\ddot{\theta} + a\dot{\theta}\dot{\varphi} = 0, \qquad (a^2+b^2)\ddot{\varphi} - a\rho\dot{\theta}^2 = -gb.$$

Trajectories of the contact point - $\mu \leq 0$



movement: $\dot{u}_x\ddot{u}_y - \ddot{u}_x\dot{u}_y = 0, \quad (a^2+b^2)\ddot{u}_z = -gb^2,$

constraint: $\dot{u}_x^2 + \dot{u}_y^2 = \dfrac{a^2}{b^2}\dot{u}_z^2.$

3.3. Second order linear constraints

Non-holonomic constraint specifying pseudo-precession:

$$(\omega_\xi \dot{\omega}_\eta - \dot{\omega}_\xi \omega_\eta) + \omega_\zeta(\omega_\xi^2 + \omega_\eta^2) - \lambda(\omega_\xi^2 + \omega_\eta^2)^{3/2} = 0$$

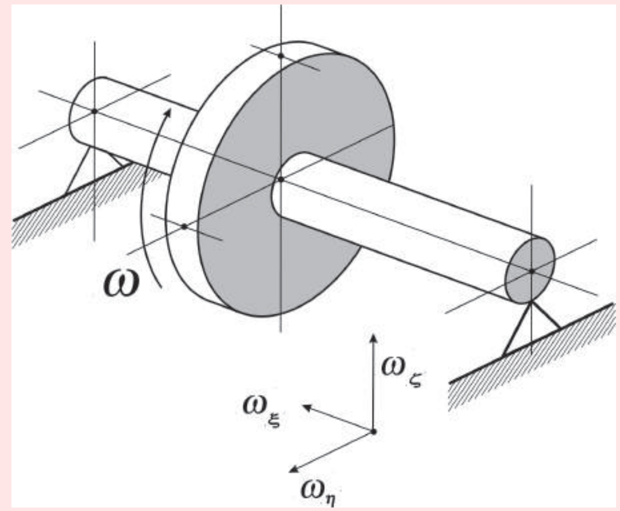
$$(\ddot{\psi}\dot{\theta} - \ddot{\theta}\dot{\psi}) \sin\theta + (2\dot{\psi}\dot{\theta}^2 + \dot{\psi}^3 \sin^2\theta) \cos\theta + \lambda(\dot{\varphi}^2 + \dot{\theta}^2)^{3/2} = 0$$

Euler angles:

$$\begin{aligned} \omega_\xi &= \dot{\psi} \sin\theta \sin\varphi \ + \dot{\theta} \cos\varphi, \\ \omega_\eta &= \dot{\psi} \sin\theta \cos\varphi \ - \dot{\theta} \sin\varphi, \\ \omega_\zeta &= \dot{\psi} \cos\theta + \dot{\varphi} \end{aligned}$$

When $\dot{\varphi}, \dot{\psi}$ are constants and $\dot{\theta}$ vanishes, the regular precession emerges. Non-trivial variable $\dot{\varphi}, \dot{\psi}, \dot{\theta}$ lead to precession. Then it is necessary to investigate the rate of stability of the precession phenomenon.

Rotating shaft - pseudo-precession.



3.4. Third order linear constraints

Lagrange function:

$$\mathcal{T} = \frac{1}{2} m(\dot{u}_x^2 + \dot{u}_y^2) + \lambda L$$

Lagrange equations:

$$m\ddot{u}_x = Q_x + \lambda L u_x, \quad m\ddot{u}_y = Q_y + \lambda L u_y,$$

Q_x, Q_y - external forces acting along axes x, y.

Curvature - planar trajectory:

$$\varrho(t) \;=\; \frac{\dot{u}_x \ddot{u}_y - \ddot{u}_x \dot{u}_y}{(\dot{u}_x^2 + \dot{u}_y^2)^{3/2}}$$

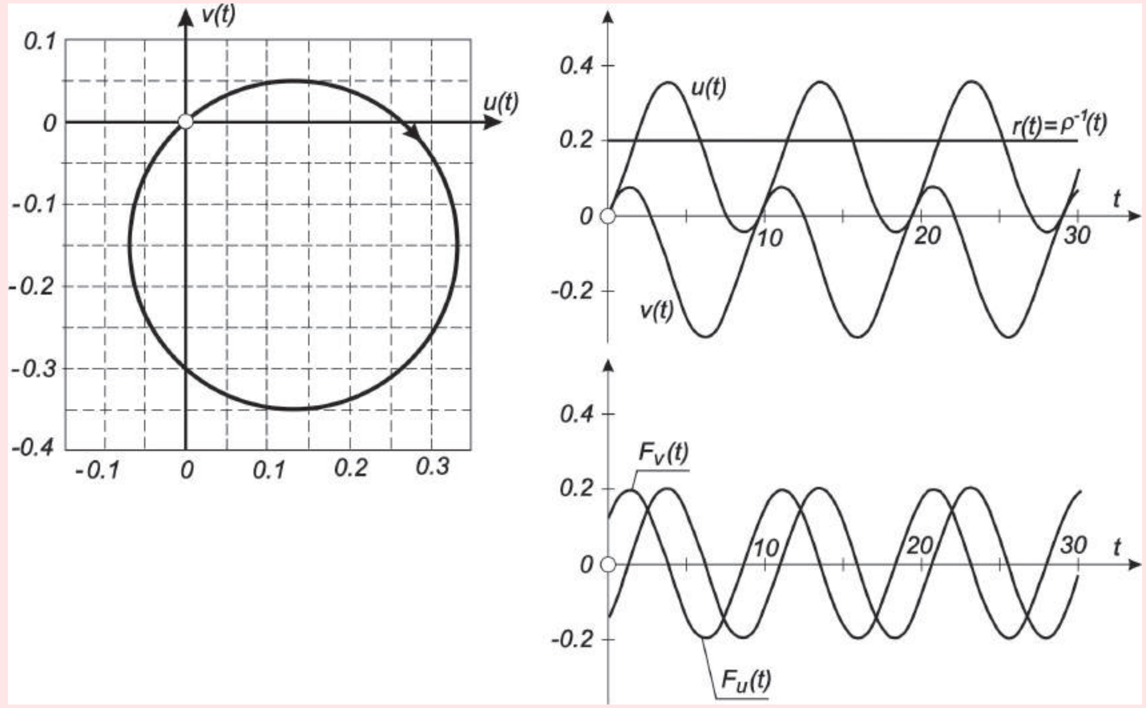
Curvature time derivative:

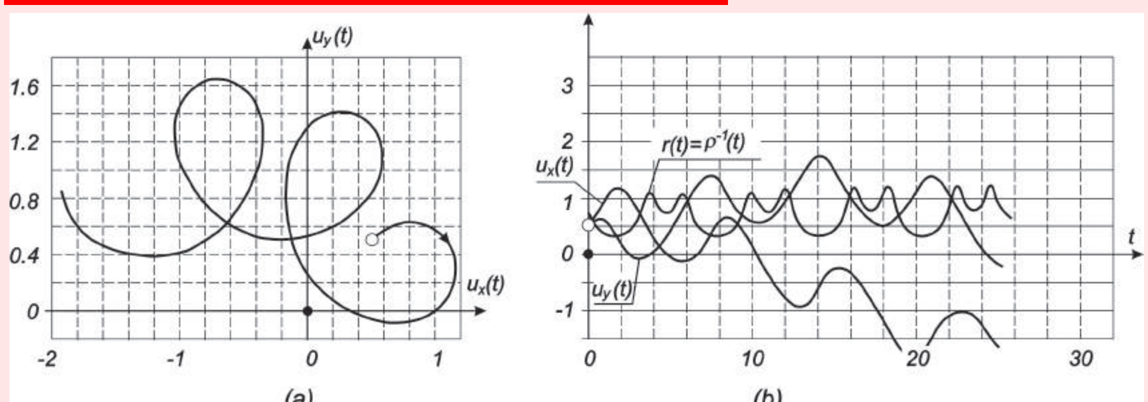
$$\dot{\varrho}(t) \;=\; \frac{\dot{u}_x \dddot{u}_y - \dddot{u}_x \dot{u}_y}{(\dot{u}_x^2 + \dot{u}_y^2)^{3/2}} - 3\varrho(t)\, \frac{\dot{u}_x \ddot{u}_x + \dot{u}_y \ddot{u}_y}{\dot{u}_x^2 + \dot{u}_y^2}$$

$\dot{\varrho}(t), \varrho(t)$ - prescribed functions of time.

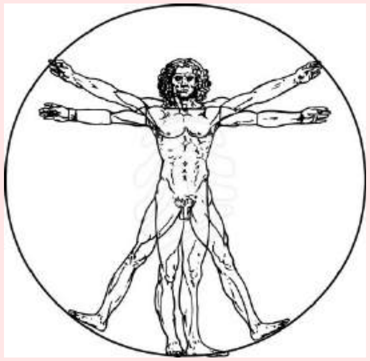
$$m(\ddot{u}_x \dot{u}_x + \ddot{u}_y \dot{u}_y) \;=\; Q_x \dot{u}_x + Q_y \dot{u}_y,$$

$$(\dot{u}_x \dddot{u}_y - \dddot{u}_x \dot{u}_y)(\dot{u}_x \ddot{u}_y - \ddot{u}_x \dot{u}_y) = -\varrho^2 (\dot{u}_x^2 + \dot{u}_y^2)^3 + 3\varrho\dot{\varrho}(\dot{u}_x \ddot{u}_x + \dot{u}_y \ddot{u}_y).$$





4. CONCLUSION - CREED OF RATIONAL MECHANICS



- **La meccanica e il paradiso delle scienze matematiche, perché con quella si viene al frutto matematico.**

  Mechanics is the paradise of the mathematical sciences, because with it we come to the most beautiful fruits of mathematical knowledge.

- **Si come il mangiare sanza voglia fia dannoso alla salute, cosi lo studio sanza desiderio guasta la memoria, e no' ritiene cosa ch'ella pigli.**

  Like eating tasteless is harmful to health, so study without desire spoils the memory, and it cannot keep you take.

Leonardo da Vinci (1452 - 1519)

*****   Thank you for your patience   *****